A Long Pseudo-comparison of Premice in $L[x]$

Farmer Schlutzenberg

Abstract

A significant open problem in inner model theory is the analysis of $\text{HOD}^L[x]$ as a strategy premouse, for a Turing cone of reals $x$. We describe here an obstacle to such an analysis. Assuming sufficient large cardinals, for a Turing cone of reals $x$ there are premice $M, N$ in $L[x]$, and countable in $L[x]$, such that the pseudo-comparison of $L[M]$ with $L[N]$ succeeds, is in $L[x]$, and lasts exactly $\omega^L$ stages. Moreover, we can take $M = M_1 | (\delta^+)^{M_1}$ where $M_1$ is the minimal iterable proper class inner model with a Woodin cardinal, and $\delta$ is that Woodin. We can take $N$ such that $L[N]$ is $M_1$-like and short-tree-iterable.

1 Introduction

A central program in descriptive inner model theory is the analysis of $\text{HOD}^W$, for transitive models $W$ satisfying $\text{ZF} + \text{AD}^+$; see [8], [6], [2], [5]. For the models $W$ for which it has been successful, the analysis yields a wealth of information regarding $\text{HOD}^W$ (including that it is fine structural and satisfies GCH), and in turn about $W$.

Assume that there are $\omega$ many Woodin cardinals with a measurable above. A primary example of the previous paragraph is the analysis of $\text{HOD}^L(\mathbb{R})$. Work of Steel and Woodin showed that $\text{HOD}^L(\mathbb{R})$ is an iterate of $M_\omega$ augmented with a fragment of its iteration strategy (where $M_n$ is the minimal iterable proper class inner model with $n$ Woodin cardinals). The addition of the iteration strategy does not add reals, and so the $\text{OD}^L(\mathbb{R})$ reals are just $\mathbb{R} \cap M_\omega$. The latter has an analogue for $L[x]$, which has been known for some time: for a cone of reals $x$, the $\text{OD}^L(x)$ reals are just $\mathbb{R} \cap M_1$. Given this, and further analogies between $L(\mathbb{R})$ and $L[x]$ and their respective HODs, it is natural to ask whether there the full $\text{HOD}^L[x]$ is an iterate of $M_1$, adjoined with a fragment of its iteration strategy. Woodin has conjectured that this is so for a cone of reals $x$; for a precise statement see [3, 8.23]. Woodin has proved approximations to this conjecture. He analyzed $\text{HOD}^{L[x,G]}$, for a cone of reals $x$, and $G \subseteq \text{Coll}(\omega, < \kappa)$ a generic filter over $L[x]$, where $\kappa$ is the least inaccessible of $L[x]$; see [3, 8.21] and [2]. However, the conjecture regarding $\text{HOD}^L(x)$ is still open.
In this note, we describe a significant obstacle to the analysis of \( \text{HOD}^{L[x]} \).

1.1 Background We give a brief summary of some relevant definitions and facts. We assume familiarity with the fundamentals of inner model theory; see \([8],[4]\). One does not really need to know the analysis of \( \text{HOD}^{L[x,G]} \), but familiarity does help in terms of motivation; the system \( \mathcal{F} \) described below relates to that analysis. We do rely on some smaller facts from \([2], \S3\). Let us give some terminology, and recall some facts from \([2]\). We say that a premouse \( N \) is pre-\( M_1 \)-like iff \( N \) is proper class, \( 1 \)-small, and has a (unique) Woodin cardinal, denoted \( \delta^N \). (The notion \( M_1 \)-like of \([2]\) makes further demands.) Let \( P,Q \) be pre-\( M_1 \)-like. Given a normal iteration tree \( \mathcal{T} \) on \( P, \mathcal{T} \) is maximal iff \( \text{lh}(\mathcal{T}) \) is a limit and \( L[M(\mathcal{T})] \) has no \( Q \)-structure for \( M(\mathcal{T}) \) (so \( L[M(\mathcal{T})] \) is pre-\( M_1 \)-like with Woodin \( \delta(\mathcal{T}) \)). A premouse \( R \) is a (non-dropping) pseudo-normal iterate of \( P \) iff there is a normal tree \( \mathcal{T} \) on \( P \) such that either \( \mathcal{T} \) has successor length and \( R = M^\mathcal{T}_{\omega} \), the last model of \( \mathcal{T} \) (and \( [0,\omega], \mathcal{T} \) does not drop), or \( \mathcal{T} \) is maximal and \( R = L[M(\mathcal{T})] \). A premouse \( R \) is a pseudo-iterate of \( P \) iff there is \( n < \omega \) and \( (R_0,R_1,\ldots,R_n) \) such that \( R_0 = P \) and \( R_n = R \) and each \( R_{i+1} \) is pre-\( M_1 \)-like and is a pseudo-normal iterate of \( R_i \). A pseudo-comparison of \( (P,Q) \) is a pair \((\mathcal{T},\mathcal{U})\) of normal padded iteration trees of equal length, formed according to the usual rules of comparison, such that either \((\mathcal{T},\mathcal{U})\) is a successful comparison, or either \( \mathcal{T} \) or \( \mathcal{U} \) is maximal. A (\( z \))-pseudo-genericty iteration is defined similarly, formed according to the rules for genericity iterations making a real \( z \) generic for Woodin’s extender algebra. We say that \( P \) is normally short-tree-iterable iff for every normal, non-maximal iteration tree \( \mathcal{T} \) on \( P \) of limit length, there is a \( \mathcal{T} \)-cofinal wellfounded branch through \( \mathcal{T} \), and every putative normal tree \( \mathcal{T} \) on \( P \) of length \( \alpha + 2 \) has wellfounded last model (that is, we never first encounter an illfounded model at a successor stage). If \( P|\delta^P \in HC^{L[x]} \), then normal short-tree-iterability is absolute between \( L[x] \) and \( V \). If \( P,Q \) are normally short-tree-iterable then there is a pseudo-comparison \((\mathcal{T},\mathcal{U})\) of \( (P,Q) \), and if \( \mathcal{T} \) has a last model then \([0,\omega],\mathcal{T} \) does not drop, and likewise for \( \mathcal{U} \).

By Turing determinacy we mean the statement that every set of Turing degrees either contains or is disjoint from a cone.

1.2 The \text{HOD} of \( L[x] \) It has been suggested\(^1\) that one might analyze \( \text{HOD}^{L[x]} \) using an \( \text{OD}^{L[x]} \) directed system \( \mathcal{F} \) such that:

- the nodes of \( \mathcal{F} \) are pairs \((N,s)\) such that \( s \in OR^{<\omega} \) and \( N \) is a normally short-tree-iterable, pre-\( M_1 \)-like premouse with \( N|\delta^N \in HC^{L[x]} \),
- for \((P,t),(Q,u) \in \mathcal{F} \), we have \((P,t) \leq_{\mathcal{F}} (Q,u) \) iff \( t \subseteq u \) and \( Q \) is a pseudo-iterate of \( P \), and
- \((M_1,\emptyset) \in \mathcal{F} \).

If such systems existed, satisfying some further requirements regarding the sets \( s \), strengthening the iterability requirements, and including countable directedness (for each fixed \( s \)), then there would have been a reasonable scenario for analyzing \( \text{HOD}^{L[x]} \), making use of Neeman’s genericity iterations.\(^2\)

The primary difficulty in analyzing \( \text{HOD}^{L[x]} \) in this manner is in arranging that \( \mathcal{F} \) be directed, even finitely. For this, it seems most obvious to try to arrange that \( \mathcal{F} \) be closed under pseudo-comparison of pairs. However, we show here that, given sufficient large cardinals, there is a cone of reals \( x \) such that if \( \mathcal{F} \) is as above, then \( \mathcal{F} \) is not closed under pseudo-comparison of pairs.
The proof proceeds by finding a node \((N, \emptyset) \in \mathbb{F}\) such that, letting \((\mathcal{T}, \mathcal{W})\) be the pseudo-comparison of \((M_1, N)\), then \(\mathcal{T}, \mathcal{W}\) are in fact pseudo-genericity iterations of \(M_1, N\) respectively, making reals \(y, z \in L[x]\) generic, where \(\omega_1^{L[y]} = \omega_1^{L[z]} = \omega_1^{L[x]}\). Letting \(W\) be the output of the pseudo-comparison, we will have \(W[\delta^W] \in L[x]\), so \(\omega_1^{W[z]} = \omega_1^{L[x]}\), which implies that \(\delta^W = \omega_1^{L[x]}\), so \((W, \emptyset) \notin \mathcal{F}\). We now proceed to the details.

2 The Pseudo-comparison

For a formula \(\varphi\) in the language of set theory (LST), \(\zeta \in \text{OR}\), and \(x \in \mathbb{R}\), let \(A^x_{\varphi, \zeta}\) be the set of all \(M \in \text{HC}_{L[x]}\) such that \(L[x] \models \varphi(\zeta, M)\), and \(L[M]\) is a normally short-tree-iterable pre-\(M_1\)-like premouse and \(M = L[M][\delta^{L[M]}]\).

Note that \(\varphi\) does not use \(x\) as a parameter. So by absoluteness of normal short-tree-iterability (between \(L[x]\) and \(V\), for elements of \(\text{HC}_{L[x]}\)), \(A^x_{\varphi, \zeta}\) is OD\(_{L[x]}\). So \(A^x_{\varphi, \zeta}\) is a collection of premice like those involved in the system \(\mathcal{F}\) (restricted to their Woodins).

**Theorem** Assume Turing determinacy and that \(M^x_1\) exists and is fully iterable. Then for a cone of reals \(x\), for every formula \(\varphi\) in the LST and every \(\zeta \in \text{OR}\), if \(M_1[\delta^{M_1}] \in A^x_{\varphi, \zeta}\) then there is \(R \in A^x_{\varphi, \zeta}\) such that the pseudo-comparison of \(M_1\) with \(L[R]\) has length \(\omega_1^{L[x]}\).

**Proof** Suppose not. Then we may fix \(\varphi\) such that for a cone of \(x\), the theorem fails for \(\varphi, x\). Fix \(z\) in this cone with \(z \geq_T M^x_1\). Let \(\mathcal{W}\) be the \(z\)-genericity iteration on \(M_1\) (making \(z\) generic for the extender algebra), and \(Q = M_{\mathcal{W}}\). By standard arguments (see [21]), \(Q[z] = L[z]\).

\[
\text{ih}(\mathcal{W}) = \omega_1^{\delta^\mathcal{W}[z]} + 1 = \delta^Q + 1,
\]

\[
Q[\delta^Q] = M(\mathcal{W} \upharpoonright \delta^\mathcal{W}), \quad \text{and} \quad \mathcal{T} = \text{def } \mathcal{W} \upharpoonright \delta^Q \text{ is the } z\text{-pseudo-genericity iteration of } M_1\text{, and } \mathcal{T} \in L[z].
\]

Let \(\mathbb{B}\) be the extender algebra of \(Q\) and let \(\mathbb{P}\) be the finite support \(\omega\)-fold product of \(\mathbb{B}\). For \(p \in \mathbb{P}\) and \(i < \omega\) let \(p_i\) be the \(i\)th component of \(p\). Let \(G \subseteq \mathbb{P}\) be \(Q\)-generic, with \(z_0 = z\) where \(x = \text{def } (z_i)_{i < \omega}\) is the generic sequence of reals. Then

\[
Q[G] = Q[x] = L[x]
\]

and \(x >_T z\). Let \(\zeta \in \text{OR}\) witness the failure of the theorem with respect to \(\varphi, x\). So \(M_1[\delta^{M_1}] \in A^x_{\varphi, \zeta}\).

By [1, Lemma 3.4] (essentially due to Hjorth), \(\mathbb{P}\) is \(\delta^Q\)-cc in \(Q\), so \(\delta^Q \geq \omega_1^{L[x]}\), but \(\delta^Q = \omega_1^{L[z]}\), so \(\delta^Q = \omega_1^{L[x]}\). So it suffices to see that there is some \(R \in A^x_{\varphi, \zeta}\) such that the pseudo-comparison of \(M_1\) with \(L[R]\) has length \(\delta^Q\).

For \(e \in \omega\) and \(y \in \mathbb{R}\) let \(\Phi^e_\mathbb{P}: \omega \to \omega\) be the partial function coded by the \(e\)th Turing program using the oracle \(y\). Let \(e \in \omega\) be such that \(\Phi^e_\mathbb{P}\) is total and codes \(M_1[\delta^{M_1}]\). Let \(x\) be the \(\mathbb{P}\)-name for \(x\), and for \(n < \omega\) let \(z_n\) be the \(\mathbb{P}\)-name for \(z_n\). Let \(p \in G\) be such that \(\text{p}[\delta^\mathcal{W}] = \psi(z_0)\), where \(\psi(v)\) asserts "\(\Phi^e_\mathbb{P}\) is total and codes a premouse \(R \in A^z_{\varphi, \zeta}\) such that the \(v\)-pseudo-genericity iteration of \(L[R]\) produces a maximal tree \(\mathcal{W}\) of length \(\delta^Q\) with \(M(\mathcal{W}) = L[\mathbb{B}] \upharpoonright \delta^Q\)." In the notation of this formula,

\[
\text{p}[\delta^\mathcal{W}] = \psi(z_0), \quad \text{because } \text{p}[\delta^\mathcal{W}] = \psi(z_0).
\]
By genericity, we may fix \( q \in G \) such that \( q \leq p \) and for some \( m > 0, q_m = q_0 \). Note that \( q \models Q^G \models \psi(\check{z}_m) \).

Let \( \hat{R}_0 \) be the \( P \)-name for the premouse coded by \( \Phi_{\check{z}_1}^{\check{z}_0} \) (or for \( \emptyset \) if this does not code a premouse). Also let \( \check{z}_0', \check{z}_1' \) be the \( B \times B \)-names for the two \( B \times B \)-generic reals (in order), and let \( R_i' \) be the \( B \times B \)-name for the premouse coded by \( \Phi_{\check{z}_i}' \).

We may fix \( r \leq q, r \in G \), such that

\[
r \models Q^G \models "\hat{R}_0 \neq \hat{R}_m". \tag{1}
\]

For otherwise there is \( r \leq q, r \in G \), such that \( r \models Q^G \models \hat{R}_0 = \hat{R}_m \). But since

\[
M_1|\delta^{M_1} = \hat{R}_0^G \notin Q,
\]

there are \( s, t \in B, s, t \leq r_0 \), such that

\[
(s, t) \models B \times B \models "\hat{R}_0 \neq \hat{R}_1".
\]

Therefore there are \( u, v \in B \), with \( u \leq r_0 \) and \( v \leq r_m \), such that

\[
(u, v) \models B \times B \models "\hat{R}_0 \neq \hat{R}_1".
\]

Let \( w \leq r \) be the condition with \( w_i = r_i \) for \( i \neq 0, m \), and \( w_0 = u \) and \( w_m = v \). Then

\[
w \models Q^G \models "\hat{R}_0 \neq \hat{R}_m",
\]

a contradiction.

So letting \( R = \hat{R}_m^G \), we have \( R \neq M_1|\delta^{M_1} \) and \( R \in A^{\check{z}}_{\check{z}_0} \) and \( Q|\delta^Q = M(\mathcal{U}) \), where \( \mathcal{U} \) is the \( A^{\check{z}}_{\check{z}_0} \)-pseudo-genericity iteration of \( L[R] \), and \( \text{lh}(\mathcal{U}) = \delta^Q \). We defined \( \mathcal{T} \) earlier. Let \( \mathcal{T}^+, \mathcal{U}^+ \) be the padded trees equivalent to \( \mathcal{T}, \mathcal{U}^+ \), such that for each \( \alpha \), either \( E_\alpha^{\mathcal{T}^+} \neq \emptyset \) or \( E_\alpha^{\mathcal{U}^+} \neq \emptyset \), and if \( E_\alpha^{\mathcal{T}^+} \neq \emptyset \neq E_\alpha^{\mathcal{U}^+} \) then \( \text{lh}(E_\alpha^{\mathcal{T}^+}) = \text{lh}(E_\alpha^{\mathcal{U}^+}) \).

Let \( (\mathcal{T}', \mathcal{U}') \) be the pseudo-comparison of \( (M_1, L[R]) \) (recall that \( L[R] \) is normally short-tree iterable as \( R \in A^{\check{z}}_{\check{z}_0} \)).

We claim that \( (\mathcal{T}', \mathcal{U}') = (\mathcal{T}^+, \mathcal{U}^+) \); this completes the proof. For this, we prove by induction on \( \alpha \) that

\[
(\mathcal{T}', \mathcal{U}') \models (\alpha + 1) = (\mathcal{T}^+, \mathcal{U}^+) \models (\alpha + 1).
\]

This is immediate if \( \alpha \) is a limit, so suppose it holds for \( \alpha = \beta \); we prove it for \( \alpha = \beta + 1 \). Let \( \lambda = \text{lh}(E_\beta^{\mathcal{T}^+}) \) or \( \lambda = \text{lh}(E_\beta^{\mathcal{U}^+}) \), whichever is defined. Because \( M(\mathcal{T}^+) = Q|\delta^Q = M(\mathcal{U}^+) \), the least disagreement between \( M_\beta^{\mathcal{T}^+} \) and \( M_\beta^{\mathcal{U}^+} \) has index \( \geq \lambda \), so we just need to see that \( E_\beta^{\mathcal{T}^+} \neq E_\beta^{\mathcal{U}^+} \).

So suppose that \( E_\beta^{\mathcal{T}^+} = E_\beta^{\mathcal{U}^+} \). In particular, both are non-empty. Then there is \( s \in G \) such that \( s \leq r \) (see line (1)) and \( s \models Q^G \models "E \subseteq L[R]|\hat{\lambda} \), but \( E \notin \check{V}^\gamma" \). For \( i = 0, m \), let \( \mathcal{T}_i \) be the \( \check{z}_i \)-pseudo-genericity iteration of \( L[R] \), then \( \mathcal{T}_0 \) and \( \mathcal{T}_m \) use identical non-empty extenders \( E \) of index \( \check{\lambda} \). Because

\[
s \models Q^G \models \psi(\check{z}_0) \) & \( \psi(\check{z}_m),\)
\]

also \( s \models Q^G \models "E \subseteq L[R]|\hat{\lambda} \), but \( E \notin \check{V}^\gamma" \); here \( E_\beta^{\mathcal{T}^+} \notin Q \) because \( \lambda \) is a cardinal of \( Q \). But since \( \mathcal{T}_i^G \) is computed in \( Q(\check{z}_i^G) \) (for \( i = 0, m \)) we can argue as before (as in the proof of the existence of \( r \) as in line (1)) to reach a contradiction. \( \square \)
A slightly simpler argument, using $B \times B$ instead of $P$, proves the weakening of the theorem given by dropping the parameter $\zeta$. The author does not see how to prove the full theorem using $B \times B$ instead of $P$. This is because in the argument given, $\zeta$ depends on $x$, and the choice of the conditions $p,q$ depend on $\zeta$.

Notes

1. For example, at the AIM Workshop on Descriptive inner model theory, June, 2014.

2. Woodin’s genericity iterations produce trees of length $\omega^L[x]$. Moreover, it seems that $HC^L[x]$ need not be sufficiently closed under the existence of collapse generics to allow an obvious analysis of $\text{HOD}^L[x]$ using Neeman’s genericity iterations. (Thanks to John Steel for pointing out an error that appeared in a draft of this paper, regarding this point.)

3. So if one tries to run the same argument but with $B \times B$ instead of $P$, one must first choose a generic pair of reals $x = (z_0,z_1)$, thus determining $\zeta$, but then even if we had tried to be selective about $z_1$, it seems there might not be any $q \in G$ analogous to that found in the proof using $P$. On the other hand, if there is no parameter $\zeta$ involved, we can be selective enough about $z_1$.

References


http://dx.doi.org/10.1007/978-3-662-21903-4. Zbl 0805.03042. MR 1300637 (95m:03099). 2


farmer.schlutzenberg@gmail.com
https://sites.google.com/site/schlutzenberg/home-1