Determinacy and Jónsson Cardinals in $L(R)$

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Abstract. Assume $\text{ZF} + \text{AD} + V = L(R)$ and let $\kappa < \Theta$ be an uncountable cardinal. We show that $\kappa$ is Jónsson, and that if $\text{cof}(\kappa) = \omega$ then $\kappa$ is Rowbottom. We also establish some other partition properties.

§1. Introduction. Assume $\text{AD} + V = L(R)$ and let $\kappa < \Theta$ be an uncountable cardinal. We will show that $\kappa$ is Jónsson. We also show that if $\text{cof}(\kappa) = \omega$ then $\kappa$ is Rowbottom. If $\kappa$ is regular then by Steel’s result, $\kappa$ is measurable and so Rowbottom (see [8, 8.27]). So $\kappa$ is Rowbottom iff $\text{cof}(\kappa) \in \{\omega, \kappa\}$. However, we also show that irrespective of its cofinality, $\kappa$ satisfies a partition property generalizing Rowbottomness, and also satisfies another partition property, superficially stronger than Jónssonness.

The history of these results are as follows. Kleinberg showed by pure determinacy arguments that all (uncountable cardinals) $\kappa < \aleph_\omega$ are Jónsson and that $\aleph_\omega$ is Rowbottom (see [2]). Jackson, extending some work joint with B. Löwe, showed, also by determinacy arguments, that all $\kappa < \aleph_{\omega_1}$ are Jónsson, and any $\kappa < \aleph_{\omega_1}$ of cofinality $\omega$ is Rowbottom. Woodin then announced the result that all $\kappa < \Theta$ are Jónsson, and that this could be shown using the directed system analysis of HOD. Later, motivated by Woodin’s announcement, Jackson, Ketchersid and Schlutzenberg independently proved the same result in joint work, along with the Rowbottom and other partition results, also through the directed system analysis. The proof depends on Lemma 2.7, though only makes direct use of the weaker Corollary 2.8. Our original argument established this corollary directly. Schlutzenberg proved the stronger Lemma 2.7, and we included this as it may be of independent interest.

Woodin has in fact proved that assuming $\text{ZF+AD}^+$, every uncountable cardinal $\kappa < \Theta$ is Jónsson. Here we limit ourselves to assuming $V = L(R)$.

We now give some notation and recall some definitions. For any set $X$ and $n < \omega$, $[X]^n$ denotes the set of subsets of $X$ of cardinality $n$, and $[X]^{<\omega}$ the finite...
subsets. We let \( \|A\| \) denote the cardinality of \( A \). We use the following partition terminology. Let \( \kappa, \gamma, \delta \) be cardinals. We write
\[ [\kappa]_{\delta}^{\omega} \rightarrow [\kappa]_{\gamma}^{\omega} \]
iff for every function \( F: [\kappa]^{\omega} \rightarrow \delta \) there is \( A \subseteq \kappa \) such that
\[ \|A\| = \kappa \quad \text{and} \quad \|F^\omega[A]\| \leq \gamma. \]
We also write
\[ [\kappa]_{\lambda}^{\omega} \rightarrow [\kappa]_{\gamma}^{\omega} \]
iff for each cardinal \( \lambda < \delta \)
\[ [\kappa]_{\lambda}^{\omega} \rightarrow [\kappa]_{\gamma}^{\omega} \]
for each cardinal \( \lambda < \delta \). The notation \[ [\kappa]_{n}^{\omega} \rightarrow [\kappa]_{n}^{\omega} \]
ext, etc, is defined similarly.

Let \( \kappa \) be an uncountable cardinal. Recall that:
\( \kappa \) is Rowbottom iff
\[ [\kappa]_{\omega}^{\omega} \rightarrow [\kappa]_{\omega}^{\omega} \]
\( \kappa \) is Jónsson iff for every
\[ [\kappa]^{\omega} \rightarrow [\kappa]^{\omega} \]
there is \( A \subseteq \kappa \) such that
\[ \|A\| = \kappa \quad \text{and} \quad F^\omega[A]^{\omega} \neq \kappa. \]

\[ \text{§2. Main Results.} \]
We now state our main results.

2.1. Theorem. Assume \( \text{AD} + V = L(\mathbb{R}) \). Let \( \kappa < \Theta \) be an uncountable cardinal. Then:
(a) If \( \text{cof}(\kappa) = \omega \) then \( \kappa \) is Rowbottom.
(b) In fact, in general, \[ [\kappa]^{\omega} \rightarrow [\kappa]_{\omega}^{\omega} \quad \text{and} \quad [\kappa]_{\omega}^{\omega} \rightarrow [\kappa]_{\omega}^{\omega} \]
(c) \( \kappa \) is Jónsson.
(d) In fact, let \( \lambda \) be a cardinal such that \( \omega_1 \leq \lambda \leq \kappa \). Let
\[ F: [\kappa]^{\omega} \rightarrow \lambda. \]

Then there is \( A \subseteq \kappa \) such that:
- \[ \|A\| = \kappa; \]
- \[ \|\lambda \setminus F^\omega[A]\| \leq \lambda; \]
- in fact, \( \lambda \setminus F^\omega[A]^{\omega} \) contains a club subset of \( \lambda \) of cardinality \( \lambda \).

As a corollary to the proof of Theorem 2.1, we obtain a simultaneous partition property:

2.2. Theorem. Assume \( \text{AD} + V = L(\mathbb{R}) \). Let \( \kappa < \Theta \) be an uncountable cardinal, \( \gamma_1, \lambda_1 < \kappa \) and \( \gamma_2, \lambda_2 < \text{cof}(\kappa) \). For each \( i \in \{1, 2\} \), let \( \{F^i_\alpha\}_{\alpha < \gamma_i} \) be such that for each \( \alpha < \gamma_i \) we have
\[ F^i_\alpha: [\kappa]^{\omega} \rightarrow \lambda_i. \]

Then there is \( A \subseteq \kappa \) such that \[ \|A\| = \kappa \]
\[ \forall \alpha < \gamma_1 \left( \|F^1_\alpha[A]\| \leq \text{cof}(\kappa) \right); \]
\[ \forall \alpha < \gamma_2 \left( \|F^2_\alpha[A]\| \leq \omega \right). \]

By a standard argument, Theorem 2.2 easily implies the following “two-cardinal” result.
2.3. **Corollary.** Assume $\text{AD} + V = L(\mathbb{R})$. Let $\kappa < \Theta$ be a cardinal, and suppose $\omega \leq \lambda_1 < \text{cof}(\kappa) \leq \lambda_2 < \kappa$. Then $(\kappa, \lambda_2, \lambda_1) \rightarrow (\kappa, \text{cof}(\kappa), \omega)$. That is, for any first order structure $M = (\kappa, \mathcal{R}, \lambda_2, \lambda_1)$ with universe $\kappa$, countably many predicates $\mathcal{R}$, and one-place predicates $\lambda_1, \lambda_2$, there is $X < M$, with $X$ having universe $A \subseteq \kappa$ with $\|A\| = \kappa$, and $\|A \cap \lambda_1\| \leq \text{cof}(\kappa)$, and $\|A \cap \lambda_1\| \leq \omega$.

**Proof.** Let $G$: $[\kappa]^{<\omega} \rightarrow \kappa$ be a Skolem function for $M$. (Take $G$ such that whenever $B \subseteq \kappa$ has limit ordertype, then $G^B[\kappa]^{<\omega} \prec M$.) Let $F_i$: $[\kappa]^{<\omega} \rightarrow \lambda_i + 1$ be defined by $F_i(b) = \min(G(b), \lambda_i)$. If $A \subseteq \kappa$ witnesses Theorem 2.2 with respect to $F_1, F_2$ then $X = G^A[A]^{<\omega}$ is as required.

2.4. **Remark.** The partition properties in Theorem 2.1 are optimal in certain ways. The property $[\kappa]^{1,\omega}_\kappa \rightarrow [\kappa]^{1,\omega}_1$ is false for any $\lambda < \kappa$ and $[\kappa]^{1,\omega}_{\text{cof}(\kappa)} \rightarrow [\kappa]^{1,\omega}_1$ is false for any $\lambda < \text{cof}(\kappa)$.

In Theorem 2.2, if $\kappa$ is singular, the requirement that the $F_i$’s be uniformly bounded by $\lambda_1$ is necessary. For let $\langle \gamma_\alpha \rangle_{\alpha < \text{cof}(\kappa)}$ be $\kappa$-cofinal and let $F_\alpha$: $\kappa \rightarrow \kappa$ be given by $F_\alpha(\beta) = \beta$ for $\beta < \gamma_\alpha$, and $F_\alpha(\beta) = 0$ otherwise. There is no $A$ as in Theorem 2.2 for this sequence.

Before we start the proofs, we mention a couple of related questions:

2.5. **Question.** Assume $\text{AD} + V = L(\mathbb{R})$. Do the partition properties of 2.1 hold for any $\kappa \geq \Theta$? Are there non-ordinal Jónsson cardinals? In particular, is $\mathbb{R}$ Jónsson?

Ralf Schindler suggested the following question.

2.6. **Question.** What is the consistency strength of $\text{ZF} + \text{"Every uncountable cardinal } \kappa \text{ (or } \kappa < \Theta) \text{ is Jónsson"}$?

The proof of Theorems 2.1 and 2.2 proceed through a few lemmas. We first use the directed system analysis of $\text{HOD}^{L(\mathbb{R})}$ to prove 2.7. Its proof is related to the proof of Steel’s result [8. 8.27], that if $\kappa$ is regular and uncountable, then it is measurable (under the same hypotheses).

2.7. **Lemma.** Assume $\text{AD} + V = L(\mathbb{R})$. Let $\kappa < \Theta$ be a cardinal such that $\omega_1 < \kappa$, and let $x \in \mathbb{R}$. Then $\text{HOD}_x \models \text{"there are } \kappa \text{-many measurables } < \kappa \text{"}.$

2.8. **Corollary.** Adopt the assumptions of Lemma 2.7 other than “$\kappa > \omega_1$”; assume $\kappa > \omega$. Then $\text{HOD}_x \models \text{"either } \kappa \text{ is measurable or is a limit of measurables"}$.

Working in $\text{ZFC}$, we then prove in Lemma 4.1, some partition properties for $\kappa$ such that $\kappa$ is either measurable or a limit of measurables. We then prove the main theorems by passing from $L(\mathbb{R})$ into some $\text{HOD}^{L(\mathbb{R})}_x$, applying Lemma 4.1 there.

§3. **Analysis of generators.** In this section we prove Lemma 3.9, which we need to prove 2.7. This involves an analysis of generators produced by certain iteration trees. We will deal in general with non-normal, fine iteration trees on premise, so as to give 3.9 more generally. However, for the purposes of proving 2.7 it suffices to consider only finite stacks of normal iteration trees.
We first discuss some non-standard terminology and facts related to iteration trees. For standard background, see [3] and [8].

Given a structure $\mathcal{N}$ and $\mu \in \operatorname{OR}^\mathcal{N}$, if $\mu$ is the largest cardinal of $\mathcal{N}$ then we let $(\mu^+)^\mathcal{N}$ denote $\operatorname{OR}^\mathcal{N}$.

Given a premouse $\mathcal{N}$ and a limit ordinal $\alpha \leq \operatorname{OR}^\mathcal{N}$, $\mathcal{N}|\alpha$ denotes the initial segment of $\mathcal{N}$ of height $\alpha$, and $\mathcal{N}|\alpha$ its passive counterpart. Also, $F^\mathcal{N}$ denotes the active extender predicate of $\mathcal{N}$, and $\mathcal{E}^\mathcal{N}$ the extender sequence of $\mathcal{N}$ ($\mathcal{E}^\mathcal{N}$ does not include the active extender).

We take iteration tree to be defined as in [3, §5], except that we drop condition (3) (i.e. the condition $\alpha < \beta \implies \operatorname{lh}(E_\alpha) < \operatorname{lh}(E_\beta)$), and strengthen the first clause of condition (4), to:

"if $T-\operatorname{pred}(\alpha + 1) = \beta$ then $\kappa = \operatorname{crit}(E_\alpha) < \rho$, for each $\gamma \in [\beta, \alpha)$".

(Here $\rho_\gamma$ refers to $\nu(E_\gamma)$.) As in [3, §5], $\mathcal{M}_{\alpha + 1}^\mathcal{T}$ denotes the "preimage" of $\mathcal{M}_{\alpha + 1}^\mathcal{T}$, and is some $\mathcal{N} \preceq \mathcal{M}_{\alpha + 1}^\mathcal{T}$ such that $\mathcal{P}(\alpha) \cap \mathcal{N} = \mathcal{P}(\alpha) \cap \mathcal{M}_{\alpha + 1}^\mathcal{T}$. Any tree satisfying all requirements of [3, §5] (including (3)), in fact satisfies the strengthening of the first clause of (4) given above.

3.1. REMARK. Let $T$ be an iteration tree and for $\alpha < \operatorname{lh}(T)$ let $\mathcal{M}_\alpha = \mathcal{M}_{\alpha + 1}^\mathcal{T}$, $E_\alpha = E_\alpha^\mathcal{T}$, etc. Let $\delta + 1 < \operatorname{lh}(T)$ and $\beta = T-\operatorname{pred}(\delta + 1)$. If $T$ is not normal then $E_\beta$ might not be close to $\mathcal{M}_{\delta + 1}^\mathcal{T}$, but we still have preservation of the degree $n$ fine structure between $\mathcal{M}_{\delta + 1}^\mathcal{T}$ and $\mathcal{M}_{\delta + 1}$, where $n = \deg(\delta + 1)$. For instance, $i_{\delta, \beta + 1}(p_{\mathcal{M}_{\delta + 1}^\mathcal{T}}) = p_{\mathcal{M}_{\delta + 1}}$. For see, e.g., [3, 4.3.4.4]. Also, letting $\mu = \operatorname{crit}(E_\delta)$,

$$\mathcal{M}_{\delta + 1}^\mathcal{T} = \mathcal{M}_{\delta + 1}^{(\mu^+)} = \mathcal{M}_{\delta + 1}^{(\mu^+)^{\mathcal{M}_{\delta + 1}^\mathcal{T}}}.$$ 

3.2. REMARK. Let $T, \delta, \beta$ be as above and $\mu = \operatorname{crit}(E_\delta)$. Let $\gamma \in [\beta, \delta)$ be such that $\operatorname{lh}(E_\gamma)$ is minimal. Then we have (a), (b), (c) below:

(a) $(\mu^+)^{\mathcal{M}_{\delta + 1}^\mathcal{T}} \leq \operatorname{lh}(E_\gamma)$ for each $\epsilon \in [\beta, \delta)$.

(b) In fact:

- For each $\epsilon \in [\beta, \gamma)$, $E_\epsilon$ is on $\mathbb{E}^{\mathcal{M}_\epsilon} \cap F^{\mathcal{M}_\epsilon}$.
- For each $\epsilon \in (\gamma, \delta)$, $\operatorname{lh}(E_\epsilon)$ is a cardinal of $\mathcal{M}_\epsilon$ and $\mathcal{M}_\epsilon|\operatorname{lh}(E_\epsilon) = \mathcal{M}_\beta|\operatorname{lh}(E_\gamma)$.
- If $\gamma' \in [\beta, \delta)$ and $\operatorname{lh}(E_{\gamma'}) = \operatorname{lh}(E_\gamma)$ then $\gamma = \gamma'$.

Proof of (b): This follows by induction through $[\beta, \delta]$ using coherence, etc.

Proof of (a): This follows from (b) since $E_\delta$ must measure exactly $\mathcal{P}(\mu) \cap \mathcal{M}_{\delta + 1}^\mathcal{T}$.

(c) Suppose $(\mu^+)^{\mathcal{M}_{\delta + 1}^\mathcal{T}} = \operatorname{lh}(E_\gamma)$. Then $\mathcal{M}_{\delta + 1}^\mathcal{T} = \mathcal{M}_\beta|\operatorname{lh}(E_\gamma)$, so $\mu$ is the largest cardinal of $\mathcal{M}_{\delta + 1}$ (recall that $\mathcal{M}_\beta|\operatorname{lh}(E_\gamma)$ projects strictly below $\operatorname{lh}(E_\gamma)$), $E_\gamma$ is type 2 (since $\mu < \nu(E_\gamma)$), and if also $T$ does not drop in model at $\delta + 1$ then $\beta = \gamma$.

Given a premouse $\mathcal{M}$ and $\kappa \in \operatorname{OR}^\mathcal{M}$, say $\kappa$ is finely measurable (fm) in $\mathcal{M}$ iff there is $E$ on $\mathbb{E}^\mathcal{M} \cap F^{\mathcal{M}}$ such that $\operatorname{crit}(E) = \kappa$ and $E$ is total over $\mathcal{M}$. Say $\kappa$ is almost finely measurable (afm) in $\mathcal{M}$ if $\kappa$ is fm in $\mathcal{M}$ or if $\mathcal{M}$ is active type 2 and $\kappa$ is fm in $\operatorname{Ult}_{\kappa}(\mathcal{M}, F^{\mathcal{M}})$.

Let us make some observations on this definition.

3.3. REMARK. Let $\mathcal{M}$ be a premouse and $\kappa < \operatorname{OR}^\mathcal{M}$. Then:

(a) If $\kappa$ is fm in $\mathcal{M}$ then $(\kappa^+)^{\mathcal{M}} < \operatorname{OR}^\mathcal{M}$. 


(b) If $\mathcal{M}$ is active and $\kappa$ is fn in $U = \text{Ult}_0(\mathcal{M}, F^\mathcal{M})$ and $(\kappa^+)^\mathcal{M} < \text{OR}^\mathcal{M}$ then $\kappa$ is fn in $\mathcal{M}$.

Proof of (a): If $\mathcal{N}$ is active and $\kappa = \text{crit}(F^\mathcal{N})$ then $(\kappa^+)^\mathcal{N} < \text{OR}^\mathcal{N}$.

Proof of (b): Let $E$ witness the fine measurability of $\kappa$ in $U$, and let $G$ be the normal measure segment of $E$. Then $G$ is on $\mathbb{E}^U \sim F^U$ (the first clause of the initial segment condition attains because $\nu(G) = (\kappa^+)^\mathcal{M}$ is a cardinal in $U$), and $(\kappa^+)^\mathcal{M} < \text{OR}^\mathcal{M}$ and $\text{OR}^\mathcal{M}$ is a cardinal of $U$ and $\mathcal{M}|\text{OR}^\mathcal{M} = U|\text{OR}^\mathcal{M}$, which implies $\text{lh}(G) < \text{OR}^\mathcal{M}$ and that $G$ is on $\mathbb{E}^\mathcal{M}$, giving (b).

3.4. Remark. Let $\mathcal{T}, \delta, \beta, \mu$ be as in 3.2 and assume that $\mathcal{T}$ does not drop in model at $\delta + 1$. Then $\mu$ is afm in $\mathcal{M}_\beta$.

Proof: If $\beta = \delta$ the statement is trivial, so assume $\beta < \delta$. Let $G$ be the normal measure segment of $E_\beta$. We will show that $G$ witnesses the afm of $\mu$ in $\mathcal{M}_\beta$. We have $\mathcal{M}_{\delta+1}^\ast = \mathcal{M}_\beta$. Let $\gamma$ be as in 3.2. By 3.2(a), $(\mu^+)_{\mathcal{M}_\beta} \leq \text{lh}(E_\gamma)$.

Suppose first that $(\mu^+)_{\mathcal{M}_\beta} \in \mathcal{M}_\beta$. Using 3.2(c),(b), we get that $(\mu^+)_{\mathcal{M}_\beta} < \text{lh}(E_\gamma)$ and both are cardinals in $\mathcal{M}_\delta$, so since $G$ is type 1 and $G$ is $\mathcal{M}_\beta$-total, $(\mu^+)_{\mathcal{M}_\beta} < \text{lh}(G) < \text{lh}(E_\gamma)$, so $G$ is on $\mathbb{E}^{\mathcal{M}_\beta}$, and $\mu$ is afm there.

Now suppose $\mu$ is the largest cardinal of $\mathcal{M}_\beta$. Then by (3.2(a),(c), we have that $E_\beta = F^{\mathcal{M}_\beta}$ is type 2 and $\gamma = \beta$. Like in the previous case (but considering the interval $[\beta + 1, \delta]$) then $G$ is on $\mathbb{E}^{\mathcal{M}_{\beta+1}}$, and the agreement between $\mathcal{M}_{\beta+1}$ and $U = \text{Ult}_0(\mathcal{M}_\beta, E_\beta)$ gives that $G$ is on $\mathbb{E}^U$.

3.5. Remark. Let $\pi : \mathcal{M} \rightarrow \mathcal{N}$ be $\Sigma_1$-elementary between premice $\mathcal{M}, \mathcal{N}$, and let $\tau \in \text{OR}^\mathcal{M}$. Suppose that if $\tau$ is the largest cardinal of $\mathcal{M}$ then $\pi(\tau)$ is likewise in $\mathcal{N}$. Then $\tau$ is afm in $\mathcal{M}$ iff $\pi(\tau)$ is afm in $\mathcal{N}$.

Proof: First assume that $(\tau^+)^\mathcal{M}$ exists in $\mathcal{M}$. By $\Sigma_1$-elementarity, $\pi((\tau^+)^\mathcal{M}) = (\pi(\tau^+))^\mathcal{N}$. So by 3.3 we need only consider fine measurability in $\mathcal{M}, \mathcal{N}$. Now $\tau$ is fn in $\mathcal{M}$ iff there is $E$ on $\mathbb{E}^{\mathcal{M}} \sim F^{\mathcal{M}}$ such that $\text{crit}(E) = \tau$ and $(\tau^+)^\mathcal{M} < \text{lh}(E)$. This property is $\Sigma_1$ in the parameter $(\tau^+)^\mathcal{M}$, and it reflects to $\pi(\tau)$ in $\mathcal{N}$.

Now suppose that $\tau$ is the largest cardinal of $\mathcal{M}$, so $\pi(\tau)$ is the largest cardinal of $\mathcal{N}$. So assume $\mathcal{M}, \mathcal{N}$ are type 2 and consider fine measurability in $U^\mathcal{M} = \text{Ult}_0(\mathcal{M}, F^\mathcal{M})$ and $U^\mathcal{N} = \text{Ult}_0(\mathcal{N}, F^\mathcal{N})$. Let $\psi : U^\mathcal{M} \rightarrow U^\mathcal{N}$ be given by $\pi$ and the shift lemma. Then $\psi$ is $\Sigma_1$-elementary and $\psi(\tau) = \pi(\tau)$, so the statement reduces to the previous case. (If $U^\mathcal{M}, U^\mathcal{N}$ are not wellfounded then the previous case doesn’t literally apply, but the first order properties of $U^\mathcal{M}, U^\mathcal{N}, \psi$ are sufficient.)

3.6. Definition. Let $\pi : \mathcal{M} \rightarrow \mathcal{N}$ be $\Sigma_1$-elementary between premice and $\gamma \in \text{OR}^\mathcal{N}$, such that $\gamma < \sup \pi^{\text{OR}^\mathcal{M}}$.

We say $\gamma$ is a generator (relative to $\pi$) iff $\gamma \neq \pi(f)(a)$ for any $f \in \mathcal{M}$ and $a \in \gamma^<\omega$.

We say that $\mathcal{N}$ has the hull property at $\gamma$ (relative to $\pi$) iff $\mathcal{P}(\gamma)^\mathcal{N} \subseteq H$, where $H$ is the transitive collapse of $\text{Hull}^\mathcal{N}_0(\gamma \cup \pi^{\text{OR}^\mathcal{M}})$.

3.7. Remark. Let $\pi : \mathcal{M} \rightarrow \mathcal{N}$ be $\Sigma_1$-elementary and $\gamma \in \text{OR}^\mathcal{N}$. Then we have (a), (b), (c), (d) below.
(a) If \( \pi(\beta) > \gamma \), then \( \gamma \) is a generator iff \( \gamma \neq \pi(f)(a) \) for all \( f: \beta^{<\omega} \to \beta \) and \( a \in \gamma^{<\omega} \).

(b) Suppose also \( \sigma: \mathcal{N} \to \mathbb{Q} \) is \( \Sigma_1 \)-elementary. Then \( \gamma \) is a generator for \( \pi \) iff \( \sigma(\gamma) \) is a generator for \( \sigma \circ \pi \).

Proof of (b): Let \( f \in \mathcal{M} \), and \( \gamma \neq \pi(f)^{\mathcal{N}} \). This lifts under \( \sigma \), and vice versa.

(c) \( \mathcal{N} \) has the hull property at \( \gamma \) iff for every \( A \in \mathcal{P}(\gamma)^{\mathcal{N}} \), there is \( f \in \mathcal{M} \) and \( a \in \gamma^{<\omega} \) such that \( \pi(f)(a) \cap \gamma = A \).

(d) If \( \mathcal{N} \) is sufficiently iterable, has the hull property at \( \gamma \), and \( H \) is the transitive collapse of \( \text{Hull}_0(\gamma \cup \pi^\mathcal{M}) \), then \( H \cap (\gamma^+) = \mathcal{N} \cap (\gamma^+) \).

Proof of (d): Use condensation.

(However, it can happen that \( H \cap (\gamma^+) \neq \mathcal{N} \cap (\gamma^+) \). For example, suppose \( \mathcal{M} \) is type 2, \( \gamma \) is the largest cardinal of \( \mathcal{M} \) and \( \gamma \) is afm in \( \mathcal{M} \). Let \( E \in \text{Ult}_0(\mathcal{M}, \mathcal{F}^{\mathcal{M}}) \) witness the latter. Let \( \mathcal{N} = \text{Ult}_0(\mathcal{M}, E) \) and let \( \pi = i_E^{\mathcal{M}} \). Then \( \mathcal{N} \) has the hull property at \( \gamma \) and \( H = \mathcal{M} \), so \( H \) is active at \( (\gamma^+) = (\gamma^+) \), but \( \mathcal{N} \) is passive there.)

3.8. Remark. The following lemma is the main result of this section. It applies to iteration trees which aren’t necessarily normal. However, for our intended application, one may assume that \( T \) is a finite stack of normal trees \( T_0, \ldots, T_{k-1} \), and that \( \eta, \xi \) are both indices of \( T_{k-1} \).

3.9. Lemma. Let \( \mathcal{M} \) be a fine-structural premouse, \( T \) an iteration tree on \( \mathcal{M}, \xi < \text{lh}(T), n = \deg_T(\xi) \). Let \( \chi \in [0, \xi]_T \) be such that \( (\chi, \xi)_T \) has no drops in model or degree. Let \( \eta \in [\chi, \xi]_T \).

Let \( \kappa \leq \rho_\mathcal{M}^{\mathcal{M}_\xi} \) be in the range of \( i_\eta^{\mathcal{M}_\xi} \). Let \( \lambda \) be the sup of all \( \gamma < \kappa \) such that \( \gamma \) is afm in \( \mathcal{M}_\xi \). It follows that \( \lambda \in \text{rg}(i_\eta^{\mathcal{M}_\xi}) \) (to be established).

For \( \alpha \in [\eta, \xi]_T \), let \( i_\eta^{\mathcal{M}_\xi}((\kappa_\alpha, \lambda_\alpha)) = (\kappa, \lambda) \) and let \( G_\alpha \) be the set of \( i_{\chi, \alpha}^- \) generators in the interval \( [\lambda_\alpha, \kappa_\alpha] \).

(1) Suppose \( \kappa < \rho_\mathcal{M}^{\mathcal{M}_\xi} \) and that \( \mathcal{M}_\eta \) has the hull property at every point in \( [\lambda_\eta, \kappa_\eta] \), relative to \( i_{\chi, \eta}^- \). Then:

(a) \( \mathcal{M}_\xi \) has the hull property at every point in \( [\lambda_\xi, \kappa_\xi] \) relative to \( i_{\chi, \xi}^- \).

(b) For each \( \alpha \in [\eta, \xi]_T \),

\[ i_\eta^{\mathcal{M}_\xi}G_\alpha = G_\xi \cap (\{\kappa_\xi\} \cup \sup i_{\alpha, \xi}^-\kappa_\alpha) \]

(c) If \( \kappa \) is afm in \( \mathcal{M}_\xi \) then \( \{\kappa_\xi\} \cup G_\xi \cup \sup i_{\alpha, \xi}^-\kappa_\eta \) is a closed set of inaccessibles of \( \mathcal{M}_\xi \).

(d) If \( \kappa \) is not afm in \( \mathcal{M}_\xi \) then \( G_\xi = i_{\eta, \xi}^-\kappa_\eta \).

(2) Suppose \( \kappa = \rho_\mathcal{M}^{\mathcal{M}_\xi} \). Then:

(a) Either \( G_\xi = \emptyset \) or \( G_\xi = \{\kappa_\xi\} \).

(b) \( \mathcal{M}_\xi|\rho_\mathcal{M}^{\mathcal{M}_\xi} \subseteq \text{Hull}_0^{\mathcal{M}_\xi}(\lambda_\xi \cup i_{\chi, \xi}^-\kappa_\eta(\mathcal{M}_\xi||\rho_\mathcal{M}^{\mathcal{M}_\xi})) \).

Proof. We start with the following claim, then prove parts (1) and (2).

Claim 1. For \( \alpha \in [\eta, \xi]_T \), let \( \lambda'_\alpha \) be the sup of all afm’s \( \gamma \) of \( \mathcal{M}_\eta \) such that \( \gamma < \kappa_\eta \). Then \( i_{\alpha, \xi}(\lambda'_\alpha) = \lambda = \lambda_\xi \). In particular, \( \lambda_\xi \in \text{rg}(i_{\alpha, \xi}) \), and \( \lambda_\alpha = \lambda'_\alpha \).

Moreover, if \( \lambda < \kappa = \rho_\mathcal{M}^{\mathcal{M}_\xi} \) then also \( \lambda \in \text{rg}(i_{\chi, \xi}) \).
PROOF. Note that if $\gamma$ is amf in $\mathcal{M}$, then $\gamma$ is a limit cardinal of $\mathcal{M}$.

Let $\theta_\alpha$ be the largest limit cardinal $\theta$ of $\mathcal{M}_\alpha$ such that $\theta \leq \kappa_\alpha$. So $\theta_\xi = i_{\alpha,\xi}(\theta_\alpha)$ is likewise with respect to $\mathcal{M}_\xi$ and the arguments above, show that induction on $\mathcal{M}_\theta$ condition and that $\gamma < \theta$.

Finally, if $\kappa = \rho_n^{\mathcal{M}_\xi}$ and $\lambda < \kappa$, then the fact that $i_{\chi,\xi}^{\mathcal{M}_\chi}$ is cofinal in $\rho_n^{\mathcal{M}_\xi}$, and the arguments above, show that $\lambda \in \text{rg}(i_{\chi,\xi})$.

We now prove (1). So assume $\kappa < \rho_n^{\mathcal{M}_\xi}$ and the hull property hypothesis of (1). Let $(1)_\xi$ be the conjunction of (1)(a)-(1)(d) (relative to $\xi$). We proceed by induction on $\alpha \in [\eta, \xi]_T$ to prove $(1)_\alpha$. The statement $(1)_\eta$ is trivial.

We focus on the case that $\alpha = \delta + 1 > \eta$ for some $\delta$. Let $\beta = T-\text{pred}(\alpha)$. By induction, $(1)_{\beta}$ holds. Let $\mu = \text{crit}(E_{\delta})$, so $\mu < \rho_n^{\mathcal{M}_\beta}$. By 3.4 either $\mu \leq \lambda_\beta$ or $\kappa_\beta \leq \mu$.

**Case 1.** $\kappa_\beta < \mu$.

Then $(1)_\alpha$ follows from $(1)_{\beta}$ and the $\Sigma_0$-elementarity of $i_{\beta,\alpha}$.

**Case 2.** $\mu \leq \kappa_\beta$.

Fix $\gamma \in [\lambda_\alpha, \kappa_\alpha]$. We first establish that the hull property holds at $\gamma$ for $\mathcal{M}_\alpha$. If $\gamma < \mu$ then this is as in Case 1 so assume $\mu \leq \gamma$. Let $\gamma^* \in \mathcal{M}_\beta$ be least such that $i_{\beta,\alpha}(\gamma^*) \geq \gamma$. So $\mu \leq \gamma^* \in [\lambda_\beta, \kappa_\beta]$, and by $(1)_{\beta}$, $\mathcal{M}_\beta$ has the hull property at $\gamma^*$. If $\mu = \gamma$ then $\mu = \gamma^*$ and $\mathcal{P}(\mu)^{\mathcal{M}_\beta} = \mathcal{P}(\mu)^{\mathcal{M}_\alpha}$ (see 3.1), and the hull property of $\mathcal{M}_\beta$ at $\mu$ then implies it of $\mathcal{M}_\alpha$ at $\mu$. So assume $\mu < \gamma$.

Every $A \in \mathcal{P}(\gamma)^{\mathcal{M}_\alpha}$ is of the form $A = [a, f]^{\mathcal{M}_\beta}_{E_\delta}$, for some $a \in \nu(E_{\delta})^{<\omega}$ and $f: \mu^{[a]} \rightarrow \mathcal{P}(\gamma)^{\mathcal{M}_\beta}$ such that $f$ is given by a generalized $r\Sigma_n$ term over $\mathcal{M}_\beta$. In fact, $f \in \mathcal{M}_\beta$. For $\gamma^*, \mu \leq \kappa_\beta < \rho_n^{\mathcal{M}_\beta}$, so $f$ is coded by a bounded subset of $\rho_n^{\mathcal{M}_\beta}$ which is generalized $r\Sigma_n$ over $\mathcal{M}_\beta$. If $\rho_n^{\mathcal{M}_\beta} > (\kappa_\beta^{+})^{\mathcal{M}_\beta}$, it is because there is $\gamma < \rho_n^{\mathcal{M}_\beta}$ such that $\text{ran}(f) \subseteq \mathcal{M}_\beta(\gamma)$; for if $f$ is unbounded then certainly $n \geq 1$, and as in [3, p.66] one can then use $f$ to give a generalized $r\Sigma_n$ definition of a subset $W$ of $\kappa_\beta$ giving a wellorder of length $(\kappa_\beta^{+})^{\mathcal{M}_\beta}$, but then $W \in \mathcal{M}_\beta$, contradiction.

So in fact $f \in \mathcal{M}_\beta$ and $A = i_{\beta,\alpha}(f)(a)$. By the hull property at $\gamma^*$ there is $f' \in \mathcal{M}_\beta$ such that

$$f' \in \text{Hull}_0^{\mathcal{M}_\beta}(\gamma^* \cup i_{\chi,\beta}^{\mathcal{M}_\chi})$$

and such that $f'(b) \cap \gamma^* = f(b)$ for all $b \in \mu^{<\omega}$. Therefore letting $A' = i_{\beta,\alpha}(f')(a)$, we have $A' \cap \gamma = A$, and

$$A' \in \text{Hull}_0^{\mathcal{M}_\alpha}(a \cup i_{\beta,\alpha}^{\mathcal{M}_\alpha} \cup i_{\chi,\alpha}^{\mathcal{M}_\chi}).$$

Now $i_{\beta,\alpha}^{\mathcal{M}_\alpha} \subseteq \gamma$ (by choice of $\gamma^*$). Let us see that we may assume $a \subseteq \gamma$, giving $A' \in \text{Hull}_0^{\mathcal{M}_\alpha}(\gamma \cup i_{\chi,\alpha} \cup i_{\chi,\alpha}^{\mathcal{M}_\chi}).$
completing the proof of the hull property at $\gamma$.

Well, $a \subseteq \nu(E^0) < i_{\beta,\alpha}(\mu)$. If $\mu < \gamma^*$ then this gives $a \subseteq \gamma$. Otherwise $\mu = \gamma^*$. Since $\gamma^* \in [\lambda_\delta, \kappa_\beta]$ and $\mu$ is afm in $\mathcal{M}_\beta$, we therefore have $\mu = \lambda_\delta$ or $\mu = \kappa_\beta$. If $\mu = \lambda_\delta$ then $\gamma = \lambda_\alpha$ (since $\gamma \geq \lambda_\alpha$ and $\gamma \leq i_{\beta,\alpha}(\gamma^*)$), and therefore again $a \subseteq \gamma$. So suppose $\lambda_\delta < \mu = \kappa_\beta$. So $\kappa_\beta$ is a successor afm of $\mathcal{M}_\beta$. Therefore $E^0_\delta$ is an order 0 measure, and we may take $a = \{\kappa_\beta\}$. But $\gamma > \mu = \kappa_\beta$, so again $a \subseteq \gamma$, as required.

Next we examine the generators $G_\alpha$.

**Subcase 1.** $\mu \leq \lambda_\beta$.

We claim (1): $G_\alpha = i_{\beta,\alpha}^\ast G_\beta$.

Let us prove (1). By 3.7, $i_{\beta,\alpha}^\ast G_\beta = G_\alpha \cap \text{rg}(i_{\beta,\alpha})$.

If $\lambda = \kappa$, this implies (1), since then for every $\epsilon \in [\eta, \xi)_T$ we have $\lambda_\epsilon = \kappa_\epsilon$, and therefore either $G_\epsilon = \emptyset$ or $G_\epsilon = \{\kappa_\epsilon\}$.

So assume $\lambda < \kappa$. Let $\gamma \in G_\alpha$; so $\gamma \in [\lambda_\alpha, \kappa_\alpha]$. Let $\gamma^*$ be least such that $i_{\beta,\alpha}(\gamma^*) \geq \gamma$; so $\gamma^* \in [\lambda_\beta, \kappa_\beta]$. If $i_{\beta,\alpha}(\gamma^*) = \gamma$ then 3.7 implies that $\gamma^* \in G_\beta$. So assume $i_{\beta,\alpha}(\gamma^*) > \gamma$; therefore $\lambda_\beta < \gamma^* \leq \kappa_\beta$. As before, there is $f \in \mathcal{M}_\beta$ and $a \in \nu(E^0_\delta)^{<\omega}$ such that $f: \mu^{<\omega} \to \gamma^*$ and $i_{\beta,\alpha}(f)(a) = \gamma$. By the hull property at $\gamma^*$, there is $g \in \mathcal{M}_\chi$ and $b \in (\gamma^*)^{<\omega}$ such that

$$f = i_{\chi,\beta}(g)(b) \cap (\gamma^*)^2.$$

But then

$$i_{\chi,\alpha}(g)(i_{\beta,\alpha}(b))(a) = \gamma,$$

and $(a \cup i_{\beta,\alpha}(b)) \subseteq \gamma$, so $\gamma \notin G_\alpha$, contradiction. This proves (1).

Finally, suppose $\kappa$ is afm in $\mathcal{M}_\xi$; we must see that $X$ is a closed set of inaccessibles of $\mathcal{M}_\alpha$, where

$$X = \{\kappa_\alpha\} \cup G_\alpha \backslash \sup i_{\eta,\alpha}^\ast \kappa_\eta.$$

We have $\kappa_\alpha$ afm, and so inaccessible, in $\mathcal{M}_\beta$. If $\lambda = \kappa$ then $X = \{\kappa_\alpha\}$, so assume $\lambda < \kappa$. Then $i_{\beta,\alpha}$ is continuous at each $\gamma \in X'$, where

$$X' = \{\kappa_\beta\} \cup G_\beta \backslash \sup i_{\eta,\beta}^\ast \kappa_\eta,$$

since by (1)$_\beta$ every $\gamma \in X'$ is inaccessible in $\mathcal{M}_\beta$ and

$$\mu \leq \lambda_\beta < \gamma < \rho^\mathcal{M}_\beta_\beta.$$

But $X = i_{\beta,\alpha}^\ast X'$, so $X$ is closed. This completes the proof of (1)$_\alpha$ in this subcase.

**Subcase 2.** $\mu > \lambda_\beta$.

By the case and subcase hypotheses, $\mu = \kappa_\beta > \lambda_\beta$.

We claim (2): $G_\alpha = (i_{\beta,\alpha}^\ast G_\beta) \cup \{\kappa_\beta\}$.

Let us prove (2). As before, since $\lambda_\beta < \kappa_\beta$ we have that $E^0_\delta$ is type 1. Therefore $\kappa_\beta$ is the only $i_{\beta,\alpha}$-generator, and (2) then follows from the hull property for $i_{\chi,\beta}$ at $\kappa_\beta$, like in the previous subcase.

By (2),

$$G_\alpha = (G_\beta \cap \kappa_\beta) \cup \{\kappa_\beta\} \cup Y,$$
where \( Y = \{ i_{\beta, \alpha}(\kappa_\beta) \} \) if \( \kappa_\beta \in G_\beta \), and \( Y = \emptyset \) otherwise. Combined with (1)\(_\beta\), this readily gives (1)\(_\alpha\) in this subcase.

This completes this subcase, Case 2 and the successor step of the induction.

We mostly leave the case that \( \alpha \) is a limit ordinal to the reader. However, let us observe why

\[
\{ \kappa_\alpha \} \cup G_\alpha \setminus \sup i_{\eta, \alpha} " \kappa_\eta
\]

is closed. Fix \( \gamma < \kappa_\alpha \), and let \( X = G_\alpha \cap (\gamma + 1) \) and \( Y = X \setminus \sup i_{\eta, \alpha} " \kappa_\eta \). It suffices to see that \( Y \) is closed.

We claim (\( \dag \)3): for any limit \( \beta \in (\chi, \xi]_\mathcal{T} \), we have

\[
G_\beta = \bigcup_{\beta' \in (\chi, \beta]} \iota_{\beta', \beta}^{\mathcal{G}} G_{\beta'}.
\]

Indeed, (\( \dag \)3) follows readily from 3.7.

Now let \( \alpha' \in [\eta, \alpha]_\mathcal{T} \) and \( \gamma' < \kappa_{\alpha'} \) be such that \( i_{\alpha', \alpha}(\gamma') = \gamma \). Then \( X = i_{\alpha', \alpha} " X' \) where \( X' = G_{\alpha'} \cap (\gamma' + 1) \), by (\( \dag \)3) and (1)\(_\beta\) for \( \beta \in [\alpha', \alpha]_\mathcal{T} \). Let \( Y' = X' \setminus \sup i_{\eta, \alpha'} " \kappa_\eta \). Then \( Y = i_{\alpha', \alpha} " Y' \), and \( Y' \) is a closed set of non-afm inaccessibles of \( M_{\alpha'} \), and \( Y' \subseteq \rho_{\alpha'} M_{\alpha'} \), so \( i_{\alpha', \alpha} \) is continuous at each point of \( Y' \), so \( Y \) is closed.

This completes our discussion of the limit case, and so completes our proof of (1).

We now prove (2). So suppose that \( \kappa = \rho_{\alpha} M_{\xi} \). Let (2)\(_\xi\) be the conjunction of (2)\(_a\) and (2)\(_b\), relative to \( \xi \).

If \( \lambda = \kappa \), then \( \lambda_\alpha = \kappa_\alpha = \rho_{\alpha} M_{\alpha} \) for all \( \alpha \in [\eta, \xi]_\mathcal{T} \), and \( G_\alpha \) is at most \( \{ \kappa_\alpha \} \). By 3.7 we therefore have \( i_{\eta, \alpha} " G_\eta \cap G_\alpha \). The rest is trivial in this case.

Suppose \( \lambda < \kappa \). Again 3.7 gives (2)\(_a\) by Claim 1. For each \( \alpha \in [\chi, \xi]_\mathcal{T} \), we have \( \text{crit}(i_{\alpha, \xi}) < \rho_{\alpha} M_{\alpha} \) since there are no drops in degree in \( (\chi, \xi]_\mathcal{T} \). Now an induction like in the proof of (1), but simpler, shows that (2)\(_a\) holds for all \( \alpha \in (\chi, \xi]_\mathcal{T} \).

\[ \dashv \]

§4. Main proofs.

Proof of Lemma 2.7. For simplicity, we only directly prove the conclusion of 2.7 with the assumption of “\( AD + V = L(\mathbb{R}) \)” replaced by “\( M_{\xi}^\mathcal{F} \) exists and is iterable in \( \text{V}^{\text{Coll}(\omega, \mathcal{F}(\mathbb{R}))} \)”. This argument combined with the argument of [9, §7] then shows that the conclusion actually follows from “\( AD + V = L(\mathbb{R}) \)”.

Regarding the argument of [9, §7] (and with notation as there), we only need the analysis of \( (\text{HOD}(\Theta))^{\mathcal{F}(\mathbb{R})} \). (Thus, we do not need the arguments of [9] analysing \( \text{HOD} \) above \( \Theta \).)

We recall a few facts from the analysis of \( \text{HOD}^{L(\mathbb{R})} \mid \Theta \) using \( M_{\xi}^\mathcal{F} \) (see [6] or [9]). There is a directed system \( \mathcal{F} \) defined in \( L(\mathbb{R}) \), consisting of premice and iteration maps, whose direct limit is \( \text{HOD}^{L(\mathbb{R})} \mid \Theta \). For this we need consider only iteration trees which are finite stacks \( (\mathcal{T}_0, \ldots, \mathcal{T}_{n-1}) \) of normal trees \( \mathcal{T}_i \), such that

\[ 1 \text{Although we have } \rho_{\alpha} M_{\alpha} \in \text{Hull}_{\alpha}(\text{rg}(i_{\chi, \alpha}) \cup \rho_{\alpha} M_{\alpha}) \text{ when } k = n, \text{ the same might not hold when } k = 0. \text{ So } G_\alpha \neq \emptyset \text{ is possible.} \]

\[ 2 \text{This implies } AD^{L(\mathbb{R})}, \text{ but is stronger; see [6] for the proof of this and the analysis of } \text{HOD}^{L(\mathbb{R})} \text{ under this assumption.} \]
for each \( i + 1 < n \), the main branch of \( T_i \) does not drop; call such trees relevant. There is a unique strategy \( \Sigma \) for \( M^\#_\omega \) having domain the set of relevant trees on \( M^\#_\omega \), and which is an \((\omega, \omega, \omega_1 + 1)\)-strategy on its domain. Let \( G \) be generic for \( \text{Coll}(\omega, \mathcal{P}(\mathbb{R})) \) and let \( \Sigma' \) be the corresponding strategy for \( V[G] \). Then \( \Sigma \subseteq \Sigma' \); this follows from the homogeneity of the forcing.

Now in \( V[G] \) there is a stack \( \langle T_i \rangle_{i < \omega} \) of normal iteration trees such that:

- \( T_0 \) is on \( Q_0 = M^\#_\omega \).
- For each \( i < \omega \), \( T_i \in \text{HC}^V \).
- Each \( T_i \) has a non-dropping main branch and the first model of \( T_{i+1} \) is the last model of \( T_i \).
- The stack is via \( \Sigma' \) (equivalently, \( \langle T_i \rangle_{i < n} \) is via \( \Sigma \) for each \( n < \omega \)).
- Let \( Q_\gamma \) be the direct limit of the \( Q_i \)'s under the iteration maps. Let \( \delta^N_0 \) denote the least Woodin cardinal of a model \( N \). Then \( \Theta = \delta^Q_0 \) and \( V^\text{HOD}^L(\omega) = V^\Theta_\omega^\omega \).
- Let \( i < \omega \). Let \( j_i : Q_i \rightarrow Q_\omega \) be the \( \Sigma' \)-iteration map. Let \( \gamma < \delta^Q_0 \). Then \( j_i \upharpoonright (Q_i(\gamma)) \in L(\mathbb{R}) \).

Now let \( \kappa < \Theta \) be a cardinal of \( L(\mathbb{R}) \) such that \( \omega_1 < \kappa \). Let \( A \) be the set of measurables of \( \text{HOD}^L(\omega) \) below \( \kappa \). Suppose \( A \) has ordertype \( < \kappa \). Then the set of \( \text{afm}'s \) of \( Q_\omega \) below \( \kappa \) also has ordertype \( < \kappa \) (for \( Q_\omega \), every \( \text{afm} \) is finely measurable since \( Q_\omega \) is not type 2).\(^3\) Work in \( Q_\omega \). Note then that the \( \text{afm} \) limits of \( \text{afm}'s \) \( \leq \kappa \) are bounded by some \( \theta < \kappa \). Let \( X \) be the set of \( \text{afm}'s \) in the interval \( (\theta, \kappa) \). If \( X \) is bounded above by some \( \gamma < \kappa \), \( \mu = 1 \) and \( \kappa_0 = \kappa \).

Otherwise let \( (\kappa_\alpha)_{\alpha < \mu} \) enumerate \( X \), in strictly increasing order. By choice of \( \theta \), this sequence is discontinuous everywhere. In either case, \( \mu < \kappa \), and in fact by increasing \( \theta \) if need be, we will assume \( \mu < \theta < \kappa_0 \).

Now in \( V[G] \), fix \( n < \omega \) such that \( \theta, \kappa \in \text{rg}(j_n) \). Let \( j_n(\bar{\kappa}) = \kappa \).

For \( \alpha < \mu \) let \( \gamma_\alpha \) be the sup of all \( \text{afm}'s \) \( \gamma \) of \( Q_\omega \) such that \( \gamma < \kappa_\alpha \). So \( \gamma_0 \leq \theta \) and for \( \alpha > 0 \), \( \gamma_\alpha = \sup_{\beta < \alpha} \kappa_\beta \). We have \( \gamma_\alpha < \kappa_\alpha \) by choice of \( \theta \). Let \( G_\alpha \) be the set of all \( j_n \)-generators in the interval \( [\gamma_\alpha, \kappa_\alpha) \). Since \( \kappa = \theta \cup \bigcup_{\alpha < \mu} [\gamma_\alpha, \kappa_\alpha) \) then \( \theta \cup \bigcup_{\alpha < \mu} G_\alpha \) includes all generators for \( j_n \) below \( \kappa \). Therefore

\[
Q_\omega | \kappa \subseteq \text{Hull}_{\bar{\kappa}}^Q \left( \theta \cup \bigcup_{\alpha < \mu} G_\alpha \cup j_n^\omega Q_n \right),
\]

However, if \( \gamma < \kappa \) is not a \( j_n \)-generator, by 3.7(b) there is \( f : \bar{\kappa}^\omega \rightarrow \bar{\kappa} \) and \( a \in \gamma^\omega \) such that \( f \in Q_n \) and \( \gamma = j_n(f)(a) \). Therefore

\[
Q_\omega | \kappa \subseteq \text{Hull}_{\bar{\kappa}}^Q | (\bar{\kappa}^+)^Q \left( \theta \cup \bigcup_{\alpha < \mu} G_\alpha \cup j_n^\omega (Q_n | (\bar{\kappa}^+)^Q_n) \right),
\]

But \( j_n \upharpoonright (\bar{\kappa}^+)^Q_n \in L(\mathbb{R}) \), and this segment of \( j_n \) suffices to compute \( (G_\alpha)_{\alpha < \mu} \). Therefore \( L(\mathbb{R}) \) sees the previous fact. Moreover, we claim that each \( G_\alpha \) has ordertype \( \leq \omega_1 \) (in fact, exactly \( \omega_1 \)). Therefore the previous fact gives a surjection \( (Q_n \times \theta \times \mu \times \omega_1)^{< \omega} \rightarrow \kappa \in L(\mathbb{R}) \). Since \( Q_n \) is countable and \( \theta, \mu, \omega_1 < \kappa \), this shows that \( \kappa \) is not a cardinal in \( L(\mathbb{R}) \), a contradiction.

\(^3\)In fact, “measurable” implies “finely measurable” for \( Q_\omega | \bar{\kappa}^Q_n \), by [5, §4].
So fix $\alpha < \mu$. Fix $m \geq n$ such that $\alpha \in \operatorname{rg}(j_m)$ and let $i \geq m$. Let $G^i_\alpha$ be the set of $j_n,i$-generators in the interval $[\gamma^i_\alpha, \kappa^i_\alpha)$, where “superscript $i$” denotes preimage under $j_i$. We would like to apply Lemma 3.9 to deduce that $j_i^{G^i_\alpha} = G^i_\alpha \cap \sup j_i^{\kappa^i_\alpha}$. Given this, then $G_\alpha \cap \sup j_i^{\kappa^i_\alpha}$ has ordertype $< \omega_1$ (since $Q_\alpha \in \mathcal{H}C^\Gamma$). But $G_\alpha = \bigcup_{i \geq m} j_i^{G^i_\alpha}$, by 3.7. So $G_\alpha$ has ordertype $\leq \omega_1$. (In fact $G_\alpha$ has ordertype $\omega_1$, since enough normal iterates are absorbed by $(\mathcal{T}_i)_{i < \omega}$.)

So we just need to see that Lemma 3.9 applies to the iteration $(\mathcal{T}_i)_{i < \omega}$, with $\mathcal{M}_\chi = Q_n, \mathcal{M}_n = Q_m$ and the ordinal $\kappa^m_n$. We must see that $Q_m$ has the hull property, relative to $j_n,m$, at every point in $[\gamma^m_n, \kappa^m_n)$. Trivially, $Q_n$ has the hull property, relative to id, at every point in $I = [\gamma^m_n, \kappa^m_n)$. As in the proof of Lemma 3.9, an induction along on the branch $b$ leading from $Q_n$ to $Q_m$ shows that for every $\beta \in b, \mathcal{M}_\beta$ has the hull property, relative to $i_{\mathcal{Q}_n, \beta}$, at every point in $i_{\mathcal{Q}_n, \beta}(I)$. This works because $Q_n$ has no afm limits of afm’s in $I$. Therefore $Q_m$ has the hull property where required. This completes the proof.

(Our use of Lemma 3.9 can be reduced to the restricted version described in Remark 3.8. For this version can be applied inductively to each $\mathcal{T}_i$ in turn, and 3.7 can be quoted when passing to the direct limit of the stack.) \( \dashv \)

We now proceed to some ZFC results which we will apply inside $\text{HOD}^{L(R)}$ in our proof of 2.1 and 2.2. These are variants of the well known fact that under ZFC, if $\kappa$ is either a measurable or a limit of measurables, then $\kappa$ is Jónsson.

Given $n < \omega$ and measures $\mu_i$ over $X_i$ for $i < n$, we write $\prod_{i < n} \mu_i$, or $\mu_0 \times \ldots \times \mu_{n-1}$, for the standard product measure $\mu$ over $\prod_{i < n} X_i$. That is, $A \in \mu$ iff for $\mu_0$-almost all $x_0, \ldots, \mu_{n-1}$-almost all $x_{n-1}$, $(x_0, \ldots, x_{n-1}) \in A$. If each $X_i$ is a finite set of ordinals, we might blur the distinction between $(x_0, \ldots, x_{n-1})$ and $x_0 \cup \ldots \cup x_{n-1}$.

Part (a) of the following lemma is due to Prikry; see [4] and [1, 8.7]. The remaining parts are straightforward variants and have proofs similar to that result. We include the proof of all parts here for completeness.

4.1. LEMMA. Assume ZFC. Let $\kappa$ be either measurable or a limit of measurables. Then:

(a) $[\kappa]^{< \omega} \rightarrow [\kappa]^{< \omega}$.
(b) If $\operatorname{cof}(\kappa)$ is measurable then $[\kappa]^{< \operatorname{cof}(\kappa)} \rightarrow [\kappa]^{< \omega}$.
(c) Suppose either $\operatorname{cof}(\kappa) = \omega$ or $\operatorname{cof}(\kappa)$ is measurable. Let $\lambda \in [\omega_1, \kappa]$ be a cardinal. Let

$$ F : [\kappa]^{< \omega} \rightarrow \lambda. $$

Then there is $A \subseteq \kappa$ such that $|A| = \kappa$ and

$$ |\lambda \setminus F^{\omega}[A]^{< \omega}| = \lambda, $$

and in fact, $\lambda \setminus F^{\omega}[A]^{< \omega}$ contains a size $\lambda$ club subset of $\lambda$.

4.2. REMARK. Assume $\kappa$ is a limit of measurables. The proof that $\kappa$ is Jónsson (of which the proof of 4.1(c) is a variant) can easily be extended to show that if $F : [\kappa]^{< \omega} \rightarrow \kappa$ then there is $A \subseteq \kappa$ such that $|A| = \kappa$ and $\operatorname{card}(A) = \kappa$ and $\operatorname{card}(\kappa \setminus F^{\omega}[A]^{< \omega}) = \kappa$.

However, if $\operatorname{cof}(\kappa) > \omega$ and $\operatorname{cof}(\kappa)$ is not Jónsson, then it is easy to see that there is a function $F : [\kappa]^{< \omega} \rightarrow \kappa$ and a club subset $C$ of $\kappa$ of size $\operatorname{cof}(\kappa)$, such that for
every $A \subseteq \kappa$ that is cofinal in $\kappa$, we have $C \subseteq F^{\omega}[A]^{< \omega}$, and therefore $F^{\omega}[A]^{< \omega}$ is stationary.

Proof. If $\kappa$ is regular, then (a) is trivial, (b) just asserts that if $\kappa$ measurable then $\kappa$ is Rowbottom, and (c) follows from the arguments for the non-measurable case. So we assume that $\kappa$ is a singular limit of measurables.

Let $\mu = \text{cof}(\kappa) < \kappa$. Fix a strictly increasing sequence $\langle \kappa_\alpha \rangle_{\alpha < \mu}$ of measurables $< \kappa$, whose supremum is $\kappa$, with $\mu < \kappa_0$ and $\lambda < \kappa$ if $\lambda < \kappa$, and such that for each $\alpha < \mu$, $\gamma_\alpha < \kappa_\alpha$ where $\gamma_\alpha = \sup_{\beta < \alpha} \kappa_\beta$. Fix a normal measure $U_\alpha$ on each $\kappa_\alpha$, and if $\mu$ is measurable, fix a normal measure $U$ on $\mu$.

First we prove (a) and (b); initially we work on both together. Fix $\lambda < \kappa$ and $F : [\kappa]^{< \omega} \to \lambda$.

For $n < \omega$ let $T_n = n^{(\omega \setminus \{0\})}$. For each $a = \{a_0 < \ldots < a_{\|a\| - 1}\} \in [\mu]^{< \omega}$ such that $\|a\| \geq 1$ and each $t \in T_{\|a\|}$, fix a sequence $\langle X_{a,t,i} \rangle_{i < \|a\|}$ such that each $X_{a,t,i} \in U_a$ and $X_{a,t,i} \cap \gamma_a = \emptyset$, and $F$ is constant over $X_{a,t}$, where

$$X_{a,t} = \prod_{i < \|a\|} [X_{a,t,i}]^{t(i)}.$$

For each $\alpha < \mu$, let $X_\alpha = \bigcap I$ where $I$ is the set of all $X = X_{a,t,i}$ such that $X \subseteq [\gamma_\alpha, \kappa_\alpha)$. There are at most $\mu$-many such $X$, so $X_\alpha \in U_\alpha$.

We now prove (a). Let (for the proof of (a))

$$A = \bigcup_{\alpha \in \mu} X_\alpha.$$

We claim that

$$\|F^{\omega}[A]^{< \omega}\| \leq \mu,$$

as required. For if $b \in [A]^{< \omega}$ then there is a unique pair $(a,t)$ such that $b \in X_{a,t}$, but $F$ is constant over $X_{a,t}$, and there are only $\mu$-many such pairs $(a,t)$. This completes the proof of (a).

We now prove (b), so assume $\lambda < \text{cof}(\kappa)$. Let

$$G : [\mu]^{< \omega} \to \bigcup_n \left( T_n \right)^{\lambda},$$

where for any $a \in [\mu]^{< \omega}$,

$$G(a) : T_{\|a\|} \to \lambda$$

is such that $G(a)(t) = F(b)$ for some (every) $b \in X_{a,t}$. Since $\lambda < \mu$, $\lambda^\omega < \mu$, so we can fix $X \in U$ such that for each $n < \omega$, $G$ is constant over $[X]^n$.

Let (for (b))

$$A = \bigcup_{\alpha \in X} X_\alpha.$$

We claim that

$$\|F^{\omega}[A]^{< \omega}\| \leq \omega,$$
as required. For if \( b \in [A]^{<\omega} \), the value of \( F(b) \) depends only on the “type” \( t \) of \( b \). That is, \( F(b_1) = F(b_2) \) whenever there are \( a_1, a_2 \in [X]^{<\omega} \) such that \( \|a_1\| = \|a_2\| \), and \( t \in T_{\|a\|} \) such that \( b_1 \in X_{a_1,t} \) and \( b_2 \in X_{a_2,t} \). But there are only \( \omega \)-many such pairs \( (\|a\|, t) \). This completes the proof of (b).

We now prove (c). We are given \( \lambda, F \).

**Case 1.** \( \lambda < \mu \).

So \( \mu \) is measurable. Let \( A \subseteq \kappa \) witness (b). Then \( A \) works. If \( \text{cof}(\lambda) > \omega \) then this is immediate. Suppose \( \text{cof}(\lambda) = \omega \). Let \( \langle \gamma_n \rangle_{n<\omega} \) be an increasing sequence of uncountable regular cardinals, cofinal in \( \lambda \). Let \( C_n = \gamma_n \setminus ([\sup F^n[A]^{<\omega}) + 1) \). Let \( C = \bigcup_{n<\omega} C_n \). Then \( C \) is club in \( \lambda \), and is as required.

**Case 2.** \( \mu < \lambda < \kappa \) and either \( \text{cof}(\lambda) \neq \mu \) or \( \text{cof}(\lambda) = \omega \).

If \( \text{cof}(\lambda) = \omega \) then use (a) combined with the argument for (c), Case 1.

If \( \text{cof}(\lambda) > \mu \) then the result follows from (a).

So suppose \( \omega < \text{cof}(\lambda) < \mu \). Let the sets \( X_{a,t} \) and \( X_a \) be defined as in the proof of (a). For each \( n < \omega \) and \( t \in T_n \), let \( F_t : [\mu]^n \to \lambda \) be defined by \( F_t(a) = F(u) \) where \( u \in X_{a,t} \). Let \( Y_i = U_i \) be such that \( F_t^{-1}[Y_i]^n \) is bounded in \( \lambda \). Let \( Y = \bigcap_i Y_i \). Let \( A = \bigcup_{a \in Y} X_a \). Then \( F^w[A]^{<\omega} \) is bounded in \( \lambda \), so \( A \) suffices.

**Case 3.** \( \omega < \mu = \lambda < \kappa \).

So \( \mu \) is measurable. Let \( X_{a,t} \) and \( X_a \) be defined as before, and let \( F_t \) be defined as in Case 2.

Let \( X_i \in U^n \) be such that for all \( a, c \in X_i \) and \( i < n \), we have \( F_i(a) < a_i \) iff \( F_i(c) < c_i \). In fact, take \( X_i \) such that if \( F_i(a) < a_i \) for \( a \in X_i \), then \( F_i(a) = F_i(c) \) whenever \( a, c \in X_i \) are such that \( a \upharpoonright i = c \upharpoonright i \).

We can fix \( X \subseteq U \), and a sequence of functions \( \langle G^m_i \rangle_{m, i<\omega} \), where \( G^m_i : [\mu]^m \to \mu \), and for all \( a \in [X]^m \), \( G^m_i(a) \geq \text{max}(a) \), and such that for each \( n, t \) with \( t \in T_n \), there are \( m, i \), with \( m \leq n \), such that for all \( a \in [X]^n \), \( F_i(a) = G^m_i(a \upharpoonright m) \). (This includes constant functions \( G^0_i \).)

Now define a club \( C \subseteq \mu \), with strictly increasing enumeration \( \langle \delta_\alpha \rangle_{\alpha<\mu} \), as follows. Let \( \delta_0 < \mu \) be least such that \( \delta_0 \notin \text{rg}(G^0_i) \) for all \( i < \omega \). Given \( \delta_\alpha \), let \( \delta_{\alpha+1} \) be the least \( \delta \) which is closed under all functions \( G^m_i \) and \( X \cap (\delta_\alpha, \delta) \neq \emptyset \). This determines \( C \).

Let \( B = X \setminus C \). Note that \( B \) has ordertype \( \mu \) and \( G^m_i[B]^{<\omega} \cap C = \emptyset \).

Let

\[ A = \bigcup_{\alpha \in B} X_\alpha. \]

Then \( A \) has ordertype \( \kappa \) and \( F^w[A]^{<\omega} \cap C = \emptyset \), so \( A \) suffices.

**Case 4.** \( \omega < \mu < \lambda \leq \kappa \) and \( \text{cof}(\lambda) = \mu \).
Fix $(\lambda_n)_{\alpha<\mu}$, a strictly increasing, continuous sequence $\subseteq \lambda$, such that for each $\alpha$, $\lambda_{\alpha+1}$ is a cardinal. Let $W : \lambda \to \mu$ be defined by $W(\beta) = \alpha$ where $\beta \in [\lambda_n, \lambda_{\alpha+1})$. Let $G = W \circ F$. By Case 3, there is $A \subseteq \kappa$ of ordertype $\kappa$ and such that $G[A] < \omega$ is non-stationary in $\mu$. Then $\lambda \setminus F[A] < \omega$ contains a club in $\lambda$ of size $\lambda$.

**Case 5.** $\omega = \mu < \lambda = \kappa$.

We will argue similarly to Case 3. Let $T$ be the set of functions $t \in < \omega$ such that if $n = \text{lh}(t) \neq 0$ then $t(n-1) \neq 0$. Given $t \in T$, define the measure

$$U_t = \prod_{i < ||t||} U_t^{i(i)}.$$  

For each $t \in T$, fix $Y_t \in U_t$ such that there is $m < \omega$ such that $F[A] \subseteq \kappa_m$. For each $i < \omega$, fix $Y_i \in U_i$, with $Y_i \subseteq [\kappa_i-1, \kappa_i)$ (where $\kappa_{i-1} = 0$), and such that for each $t$,

$$\prod_{i < ||t||} [Y_i]^{t(i)} \subseteq Y_t.$$  

Let $I$ be the set of pairs $(m, s)$ such that $m < \omega$ and $s \in \omega^{m+1}$. There are sequences $(X_i)_{i<\omega}$ and $(H_{m,s})_{(m,s) \in I}$ such that, with

$$X_s = \prod_{i < \text{lh}(s)} [X_i]^{s(i)},$$

we have

- $X_i \in U_i$ and $X_i \subseteq Y_i$,
- $H_{m,s} : X_s \to [\kappa_{m-1}, \kappa_m)$, where $\kappa_{-1} = 0$,
- for all $u \in X_s$, $H_{m,s}(u) \geq \text{max}(u)$,
- for each $t \in T$, there is $(m, s)$ such that
  
  (i) either
  
  * $s = t \upharpoonright \text{lh}(s)$ for some $k$,
  * $s = t \setminus \langle 0, \ldots, 0 \rangle$, or
  * letting $j = \text{lh}(s) - 1$, we have $s \upharpoonright j = t \upharpoonright j$ and $s(j) < t(j)$;

  (ii) for all $u$ such that

  $$u \in \prod_{i < ||t||} [X_i]^{t(i)},$$

  we have $F(u) = H_{m,s}(u \upharpoonright l)$, where $l = \sum_{i < \text{lh}(s)} s(i)$.

This can be seen like in Case 3.

Now let $C_n \subseteq \kappa_n$ be the club of points $\gamma \in (\kappa_{n-1}, \kappa_n)$ such that for each $s \in \omega^{n+1}$ we have $H_{n,s}[\gamma] < \omega \subseteq \gamma$. There are clubs $D_n \subseteq C_n$ and sets $A_n \subseteq \kappa_n$, each of ordertype $\kappa_n$, such that $A_n \subseteq X_n \setminus D_n$. Pick closed sets $D'_n \subseteq D_n$ such that $D'_n$ is bounded in $\kappa_n$ and $D = \bigcup_{n<\omega} D'_n$ has cardinality $\kappa$.

Let $A = \bigcup_{n<\omega} A_n$. Then $A$ is as required, as $D$ is club in $\kappa$, $\|A\| = \kappa = \|D\|$ and $F[A] < \omega \cap D = \emptyset$.

We can now prove the main theorems. We only explicitly prove 2.1; an examination of its proof also yields 2.2.
Proof of Theorems 2.1, 2.2. We first prove 2.1(b),(d). Work in $L(\mathbb{R})$.

Let $\kappa \in [\omega_1, \Theta]$ be a cardinal and let $F: |\kappa|^\omega \to \kappa \leq \kappa$. Let $\mu = \text{cof}(\kappa)$ and let $f: \mu \to \kappa$ be cofinal. Fix $x \in \mathbb{R}$ such that $f, F \in \text{HOD}_x$. We have $\text{HOD}_x \models \text{ZFC}+$ “Either $\kappa$ is measurable or is a limit of measurables, and either $\mu = \omega$ or $\mu$ is measurable”, by Corollary 2.8 and [8, 8.25]. So Lemma 4.1 applies there, yielding a suitably homogeneous set $A \subseteq \kappa$. But then $A$ works in $V = L(\mathbb{R})$ also.

Part (b) also gives (a) (if $\text{cof}(\kappa) = \omega$ then $\kappa$ is Rowbottom). Part (d), in the case that $\lambda = \kappa$, gives (c) (that $\kappa$ is Jónsson). 4 5

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1Here is a slightly alternative argument for Jónssonness. Let $G: [\kappa]^{<\omega} \to \kappa$. We need a set $A \subseteq \kappa$ of ordertype $\kappa$ such that $G''[A]^{<\omega} \neq \kappa$. If $\kappa$ is regular then let $x \in \mathbb{R}$ be such that $G \in \text{HOD}_x$ and use the fact that $\kappa$ is measurable, and therefore Jónsson, in $\text{HOD}_x$. So assume $\mu = \text{cof}(\kappa) < \kappa$. If $\mu > \omega_1$ then let $\lambda = \omega_1$ (then $\lambda < \mu$); if $\mu \leq \omega_1$ then let $\lambda = \omega_2$ (then $\lambda < \kappa$ since $\kappa$ is singular). Let $F: [\kappa]^{<\omega} \to \lambda + 1$ be defined by $F(a) = \text{min}(G(a), \lambda)$. By part (b), there is $A \subseteq \kappa$ of ordertype $\kappa$ such that if $\lambda = \omega_1$ then $\|F''[A]^{<\omega}\| \leq \omega$, and if $\lambda = \omega_2$ then $\|F''[A]^{<\omega}\| \leq \mu \leq \omega_1$. In either case, it follows that $G''[A]^{<\omega} \neq \kappa$, as required.

2We didn’t actually need the full analysis of $\text{HOD}^L(\Theta)$ for the proofs of either 2.1 or 2.2. For let $\psi$ be the assertion that one of them fails. By the coding lemma, $\psi \in \Sigma^1_4$. Then, letting $\gamma$ be least such that $\mathcal{J}_\alpha(\mathbb{R}) \models \psi + \text{ZF} \models \mathcal{P}(\mathcal{P}(\mathbb{R}))$ exists”, it suffices to analyse $(\text{HOD}(\mathcal{J}_\alpha^1(\mathbb{R})))^\mathcal{J}_\alpha(\mathbb{R})$, using the argument of [7], combined with a reflection argument like that in [9, §7].